Solution to Assignment 1

1. Consider the function g in $\mathbb{R}^2$ defined by $g(x,y)=1$ whenever x,y are rational numbers and equals to 0 otherwise. Show that g is not integrable in any rectangle.

   Solution. Let P be any partition of the rectangle. By choosing tags points (x_j^*, y_k^*) where x_j^* and y_k^* are rational numbers,

$$\sum_{j,k} g(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} \Delta x_j \Delta y_k$$

   which is equal to the area of R. On the other hand, by choosing the tags so that x_j^* is irrational, $g(x_j^*, y_k^*) = 0$ so that

$$\sum_{j,k} g(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} 0 \times \Delta x_j \Delta y_k = 0 \ .$$

   Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit no matter how small their norms are. We conclude that H is not integrable.